IF-THEN RULES AND FUZZY INFERENCE

Inference

inference
\In"fer*ence\, n. [From Infer.]

1. The act or process of inferring by deduction or induction.

2. That which inferred; a truth or proposition drawn from another which is admitted or supposed to be true; a conclusion; a deduction. --Milton.

Inference is a process of obtaining new knowledge through existing knowledge.

Representation of knowledge

- To perform inference, knowledge should be represented in some form

Representation of knowledge as rules is the most popular form.

if x is A then y is B
(where A and B are linguistic values defined by fuzzy sets on universes of discourse X and Y).

- A rule is also called a fuzzy implication
- "x is A" is called the antecedent or premise
- "y is B" is called the consequence or conclusion

http://if.kaist.ac.kr/lecture/cs670/textbook/

Representation of knowledge

Examples:

- If pressure is high, then volume is small.
- If the road is slippery, then driving is dangerous.
- If an apple is red, then it is ripe.
- If the speed is high, then apply the brake a little.

http://if.kaist.ac.kr/lecture/cs670/textbook/

Knowledge as Rules

- How do you reason?
  - You want to play golf on Saturday or Sunday and you don't want to get wet when you play.
- Use rules!
  - If it rains, you get wet!
  - If you get wet, you can't play golf
- If it rains on Saturday and won't rain on Sunday
  - You play golf on Sunday!

*Fuzzy Thinking:The new Science of Fuzzy Logic, Bart Kosko

Knowledge as Rules

- Knowledge is rules
- Rules are in black-and-white language
  - Bivalent rules
- AI has so far, after over 30 years of research, not produced smart machines!
  - Because they can't yet put enough rules in the computer (use 100-1000 rules, need >100k}
  - Throwing more rules at the problem

*Fuzzy Thinking:The new Science of Fuzzy Logic, Bart Kosko

Forms of reasoning

Generalized Modus Ponens:

Premise: *x* is A'
Implication: if *x* is A then *y* is B
Consequence: *y* is B'

Where A, A', B, B' are fuzzy sets and *x* and *y* are symbolic names for objects.

Forms of reasoning

Generalized Modus Tolens:

Premise: *y* is B'
Implication: if *x* is A then *y* is B
Consequence: *x* is A'

Where A, A', B, B' are fuzzy sets and *x* and *y* are symbolic names for objects.

Fuzzy rule as a relation

if x is A then y is B
- "x is A", "y is B" – fuzzy predicates A(x), B(y)

if A(x) then B(y)
- can be represented as a relation

R(x,y): A(x) $\rightarrow$ B(y)
- where R(x,y) can be considered a fuzzy set with 2-dimentional membership function

$\mu_R(x,y)=f(\mu_A(x),\mu_B(y))$
- where f is fuzzy implication function

MIN fuzzy implication

- Interprets the fuzzy implication as the minimum operation [Mamdani].

$$R_C = A \times B = \int_{X \times Y} \mu_A(x) \wedge \mu_B(y) / (x, y)$$

where $\wedge$ is the min operator

PRODUCT fuzzy implication

- Interprets the fuzzy implication as the product operation [Larsen].

$$R_P = A \times B = \int_{X \times Y} \mu_A(x) \bullet \mu_B(y) / (x, y)$$

where $\bullet$ is the algebraic product operator

EXAMPLE OF FUZZY IMPLICATION

Fuzzy rule:
"If temperature is high, then humidity is fairly high"

Lets define:
- T – universe of discourse for temperature
- H – universe of discourse for humidity
- $t \in T$, $h \in H$ – variables for temperature and humudity
- Denote "high" as A, $A \subseteq T$
- Denote "fairly high" as B, $B \subseteq H$

Then the rule becomes:
R(t,h): if t is A then h is B or R(t,h): R(t) $\rightarrow$ R(h)

EXAMPLE OF FUZZY IMPLICATION

if we know A and B, we can find R(t,h)=A×B

t	20	30	40
$\mu_A(t)$	0.1	0.5	0.9

h	20	50	70	90
$\mu_B(h)$	0.2	0.6	0.7	1

$R_C(t, h) = A \times B$
$= \int \mu_A(t) \wedge \mu_B(h) / (t, h)$

Mamdani (min) implication

t \ h	20	50	70	90
20	0.1	0.1	0.1	0.1
30	0.2	0.5	0.5	0.5
40	0.2	0.6	0.7	0.9

EXAMPLE OF FUZZY IMPLICATION

we know $R_C(t, h)$ for fuzzy rule
"If temperature is high, then humidity is fairly high"

According to this rule, what is the humidity when "temperature is fairly high" or t is A', A'⊆T ?

t	20	30	40
$\mu_{A'}(t)$	0.01	0.25	0.81

EXAMPLE OF FUZZY IMPLICATION

We can use composition of fuzzy relations to find R(h)!

t	20	30	40
$\mu_{A'}(t)$	0.01	0.25	0.81

R(t)

t \ h	20	50	70	90
20	0.1	0.1	0.1	0.1
30	0.2	0.5	0.5	0.5
40	0.2	0.6	0.7	0.9

$R_C(t, h)$

h	20	50	70	90
$\mu_{B'}(h)$	0.2	0.6	0.7	0.81

$R(h) = R(t) \circ R_C(t, h)$

COMPOSITIONAL RULE OF INFERENCE

In order to draw conclusions from a set of rules (rule base) one needs a mechanism that can produce an output from a collection of rules. This is done using the compositional rule of inference.

Consider a single fuzzy rule and its inference
Rule: if v is A then w is C
Input: v is A′
Result: C′
$A \subset U$, $C \subset W$, $v \in U$, and $w \in C$.
The fuzzy rule is interpreted as an implication
$R: A \rightarrow C$ or $R = A \times C$
When input A′ is given to the inference system,
the output $C' = A' \circ R$

COMPOSITIONAL RULE OF INFERENCE

$C' = A' \circ R$

"o" is the composition operator. The inference procedure is called "compositional rule of inference". The inference mechanism is determined by two factors:

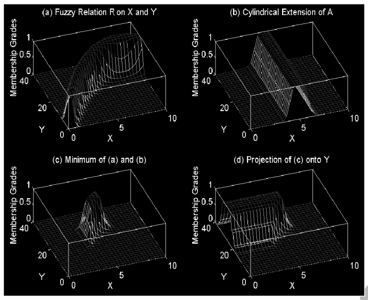
1. Implication operators:
   - Mamdani: min
   - Larsen: algebraic product
2. Composition operators:
   - Mamdani: max-min
   - Larsen: max-product

COMPOSITIONAL RULE OF INFERENCE

Compositional rule of inference can be represented graphically as a combination of cylindrical extension, intersection and projection of fuzzy sets:

1. Build a cylindrical extension of A, A(x,y)
2. Determine intersection of R(x,y) and A(x,y)
3. Build projection of R(x,y)∧A(x,y)

COMPOSITIONAL RULE OF INFERENCE



INFERENCE METHODS

There are many methods to perform fuzzy inference. Consider a fuzzy rule:

R_1: if u is A_1 and v is B_1 then w is C_1

Inputs u and v can be:

- crisp inputs. Crisp inputs can be treated as fuzzy singletons
- fuzzy sets A' and B'

MAMDANI METHOD

This method uses the minimum operation R_C as a fuzzy implication and the max-min operator for the composition.

Suppose a rule base is given in the following form:

R_i: if u is A_i and v is B_i then w is C_i, i = 1, 2, …, n

for $u \in U$, $v \in V$, and $w \in W$.

Then, $R_i = (A_i \text{ and } B_i) \rightarrow C_i$ is defined by

$$\mu_{R_i} = \mu_{(A_i \text{ and } B_i \rightarrow C_i)}(u, v, w)$$

MAMDANI METHOD

Case 1: Inputs are crisp and treated as fuzzy singletons.

$u = u_0$, $v = v_0$

$$\underbrace{\mu_{C_i'}(w)}_{\text{Inference Result}} = \underbrace{[\mu_{A_i}(u_0) \text{ and } \mu_{B_i}(v_0)]}_{\text{if …}} \rightarrow \underbrace{\mu_{C_i}(w)}_{\text{then …}}$$

Example:

if temperature is high and humidity is high then fan speed is high

How to determine the fan speed for temperature 85ºF and humidity 93%?

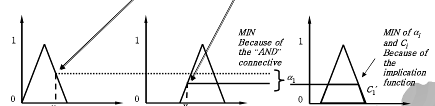
MAMDANI METHOD

Mamdani method uses min operator ($\wedge$) as fuzzy implication function ($\rightarrow$):

$$\mu_{C_i'}(w) = \alpha_i \wedge \mu_{C_i}(w)$$

$$where \ \ \alpha_i = \mu_{A_i}(u_0) \wedge \mu_{B_i}(v_0)$$

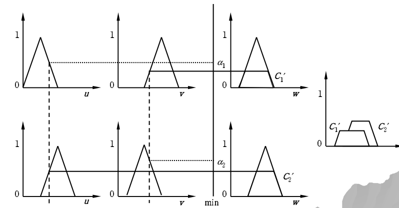
α_i is called "firing strength", "matching degree", "satisfaction degree"



MAMDANI METHOD

For multiple rules (for example, two rules R_1 and R_2):

$$\mu_{C'}(w) = \mu_{C_1'} \vee \mu_{C_2'}$$
$$= [\alpha_1 \wedge \mu_{C_1}(w)] \vee [\alpha_2 \wedge \mu_{C_2}(w)]$$

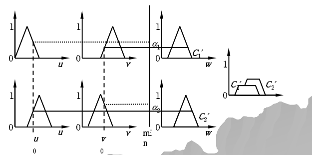


MAMDANI METHOD

In general:

$$\mu_{C'}(w) = \mathop{\vee}_{i=1}^{n}[\alpha_i \wedge \mu_{C_i}(w)] = \mathop{\vee}_{i=1}^{n} \mu_{C_i'}(w)$$

max min

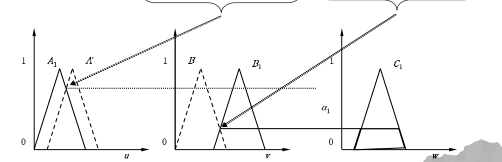


MAMDANI METHOD

Case 2: Inputs are fuzzy sets *A'*, B'

$$\mu_{C_i'}(w) = \alpha_i \wedge \mu_{C_i}(w)$$

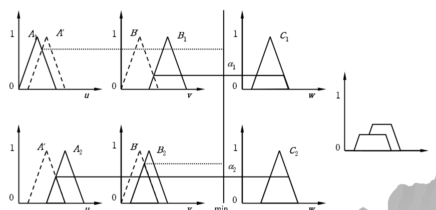
$$where\ \alpha_i = \min[\max_u(\mu_{A'}(u) \wedge \mu_{A_i}(u)),\ \max_v(\mu_{B'}(v) \wedge \mu_{B_i}(v))]$$



MAMDANI METHOD

For multiple rules, $\mu_{C'}(w) = \mathop{\vee}_{i=1}^{n}[\alpha_i \wedge \mu_{C_i}(w)] = \mathop{\vee}_{i=1}^{n} \mu_{C_i'}(w)$

$$C' = \bigcup_{i=1}^{n} C_i'$$



EXAMPLE OF MAMDANI METHOD

Let the fuzzy rule base consist of one rule:

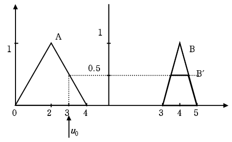
R: If *u* is A then *v* is B

where A=(0, 2, 4) and B=(3, 4, 5) are triangular fuzzy sets

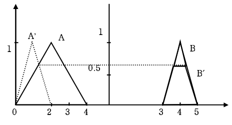
Question 1: What is the output B' if the input is a crisp value $u_0=3$?

Question 2: What is the output B' if the input is a fuzzy set A′=(0, 1, 2)?

EXAMPLE OF MAMDANI METHOD



Fuzzy inference with input u_0=3



Fuzzy inference with input A′=(0, 1, 2).

LARSEN METHOD

This method uses the product operation R_P as a fuzzy implication and the max-product operator for the composition.

Suppose a rule base is given in the following form:

R_i: if u is A_i and v is B_i then w is C_i, i = 1, 2, …, n for u ∈ U, v ∈ V, and w ∈ W.

Then, R_i = (A_i and B_i) → C_i is defined by

$$\mu_{R_i} = \mu_{(A_i\ and\ B_i \to C_i)}(u, v, w)$$

LARSEN METHOD

Case 1: Inputs are crisp and treated as fuzzy singletons.

$u = u_0$, $v = v_0$

$$\underbrace{\mu_{C_i'}(w)}_{\text{Inference result}} = \underbrace{[\mu_{A_i}(u_0) \text{ and } \mu_{B_i}(v_0)]}_{\text{if} \ldots} \rightarrow \underbrace{\mu_{C_i}(w)}_{\text{then} \ldots}$$

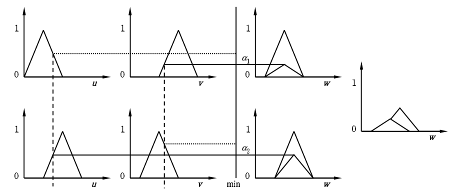
$$= [\mu_{A_i}(u_0) \wedge \mu_{B_i}(v_0)] \circ \mu_{C_i}(w)$$

$$= \alpha_i \circ \mu_{C_i}(w) \quad \text{where } \alpha_i = \mu_{A_i}(u_0) \wedge \mu_{B_i}(v_0)$$

For multiple rules: $\mu_{C'}(w) = \bigvee_{i=1}^{n} [\alpha_i \circ \mu_{C_i}(w)] = \bigvee_{i=1}^{n} \mu_{C_i'}(w)$

$$C' = \bigcup_{i=1}^{n} C_i'$$

LARSEN METHOD



Graphical representation of Larsen method with singleton input

LARSEN METHOD

Case 2: Inputs are fuzzy sets A', B'

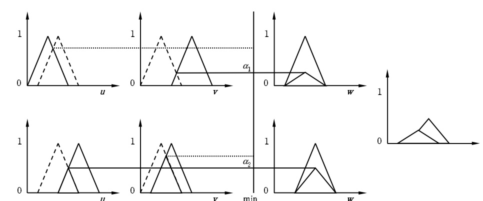
$$\mu_{C_i'}(w) = \alpha_i \circ \mu_{C_i}(w)$$
$$\text{where } \alpha_i = \min[\max_u (\mu_{A'}(u) \wedge \mu_{A_i}(u)), \max_v (\mu_{B'}(v) \wedge \mu_{B_i}(v))]$$

For multiple rules:

$$\mu_{C'}(w) = \bigvee_{i=1}^{n} [\alpha_i \circ \mu_{C_i}(w)] = \bigvee_{i=1}^{n} \mu_{C_i'}(w)$$

$$C' = \bigcup_{i=1}^{n} C_i'$$

LARSEN METHOD



Graphical representation of Larsen method with fuzzy set inputs

EXAMPLE OF LARSEN METHOD

Let the fuzzy rule base consist of one rule:

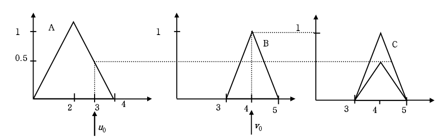
R: If *u* is A and *v* is B then *w* is C

where A=(0, 2, 4), B=(3, 4, 5) and C=(3,4,5) are triangular fuzzy sets

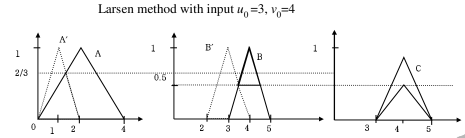
Question 1: What is the output C' if the inputs are crisp values u_0=3, v_0=4?

Question 2: What is the output C' if the inputs are fuzzy sets A′=(0, 1, 2) and B′=(2,3,4)?

EXAMPLE OF LARSEN METHOD



Larsen method with input u_0=3, v_0=4



Larsen method with input A′=(0, 1, 2), B′=(2, 3, 4).

DEFUZZIFICATION

- The output of Mamdani and Larsen inference methods is a fuzzy set!
- For practical applications a crisp value is often needed
- The process of converting a fuzzy answer into a crisp value is called defuzzification

SUMMARY

- Inference - the logical process by which new facts are derived from known facts by the application of inference rules.
- Fuzzy rules – a convenient way to represent knowledge
- A fuzzy rule can be represented as a fuzzy relation connected by a fuzzy implication function
- The fuzzy inference procedure is called the compositional rules of inference

SUMMARY

- Mamdani and Larsen methods are two very popular methods of fuzzy inference.
- There are many more inference methods that we will consider later!
- Defuzzification is needed for the results obtained through fuzzy inference.